Translations

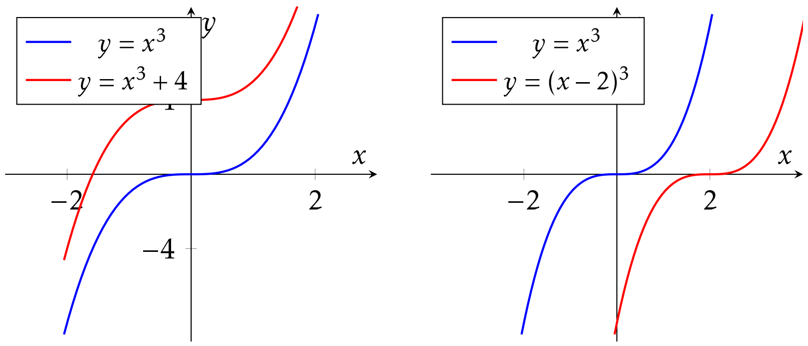
Throughout this topic, $y = f(x)$ is a known graph and we ask what happens to it when the equation is altered. The base curve here is $y = x^3$.

Fact —

$y = f(x) + a$ translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$ (y-values all increase by a)

$y = f(x + a)$ translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ (the input $x + a$ reaches each value *earlier*)

Changes *outside* f act on y, in the expected direction. Changes *inside* f act on x, in the opposite direction.



Example

The point $P(3, 7)$ lies on $y = f(x)$. Write down the image of P on:

1. $y = f(x) - 5$
2. $y = f(x + 2)$
3. $y = f(x - 1) + 4$

1. $(3, 2)$ *2.* $(1, 7)$ *3.* $(4, 11)$

Example

The graph of $y = x^2$ is translated by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Find the equation of the image in the form $y = x^2 + bx + c$.

$y = (x-3)^2 - 2 = x^2 - 6x + 7$

Textbook Exercises: SPS Course 3.2, Exercise 2 Q1–6

Stretches and Reflections

Fact —

$y = af(x)$	stretch, scale factor a, parallel to the y-axis
$y = f(ax)$	stretch, scale factor $\frac{1}{a}$, parallel to the x-axis
$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis

The inside/outside rule again: inside acts on x, opposite to expectation.

Example

The point $Q(6,-2)$ lies on $y = f(x)$. Write down the image of Q on:

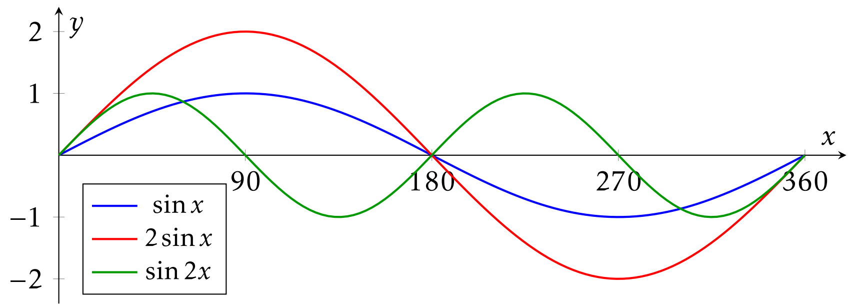
1. $y = 3f(x)$
2. $y = f(2x)$
3. $y = f(-x)$
4. $y = -f(x)$

1. $(6,-6)$ *2.* $(3,-2)$ *3.* $(-6,-2)$ *4.* $(6,2)$

Trigonometric Graphs

Example

On the same axes for $0° \leq x \leq 360°$, sketch $y = \sin x$, $y = 2\sin x$ and $y = \sin 2x$. State the number of solutions of $\sin 2x = \frac{1}{2}$ in this interval.



$y = \sin 2x$ completes two full cycles in 360°, so the line $y = \frac{1}{2}$ crosses it 4 times.

Example

Describe the transformation taking $y = \cos x$ to:

1. $y = \cos x - 3$
2. $y = \cos\left(\frac{x}{2}\right)$
3. $y = -\cos x$

1. Translation $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ 2. Stretch factor 2 parallel to the x-axis 3. Reflection in the x-axis

Textbook Exercises: SPS Course 3.2, Exercise 2 Q7–12

Recognising Transformations

Example

In each case, describe a single transformation taking the first graph to the second:

1. $y = x^2$ to $y = x^2 + 6x + 9$
2. $y = \frac{1}{x}$ to $y = \frac{3}{x}$
3. $y = 2^x$ to $y = 2^{x+1}$ — find a second, different, single transformation that also works.

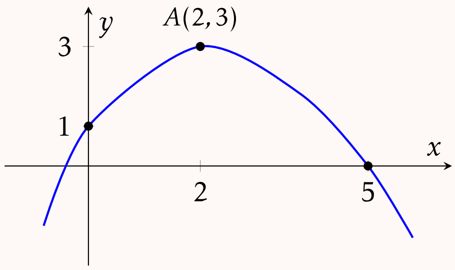
1. $y = (x+3)^2$: translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

2. Stretch factor 3 parallel to the y-axis. (Also a stretch factor $\frac{1}{3}$ parallel to the x-axis: $\frac{3}{x} = \frac{1}{x/3}$.)

3. Translation $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$; or, since $2^{x+1} = 2 \cdot 2^x$, a stretch factor 2 parallel to the y-axis.

Example

The diagram shows $y = f(x)$, which has a maximum at $A(2,3)$ and crosses the axes at $(0,1)$ and $(5,0)$.



Sketch the following, marking the images of all three points:

1. $y = f(x-3)$
2. $y = 2f(x)$
3. $y = -f(x)$
4. $y = f(-x)$

1. *Maximum* $(5,3)$; *intercepts move to* $(3,1)$ *and* $(8,0)$.
2. *Maximum* $(2,6)$; $(0,2)$, $(5,0)$ *fixed on the x-axis.*
3. *Minimum* $(2,-3)$; $(0,-1)$, $(5,0)$.
4. *Maximum* $(-2,3)$; $(0,1)$ *fixed,* $(-5,0)$.

Example

The curve $y = x^3$ is reflected in the x-axis, then translated by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$. Find the equation of the result, and the coordinates of the point where it crosses the x-axis.

$y = -x^3 + 4 = 4 - x^3$. $\quad x^3 = 4 \implies x = \sqrt[3]{4}$: *crosses at* $(\sqrt[3]{4}, 0)$.

Textbook Exercises: SPS Course 3.2, Exercise 2 (remaining questions) and Revision Exercise 3.2